An Implementable Scheme for Universal Lossy Compression

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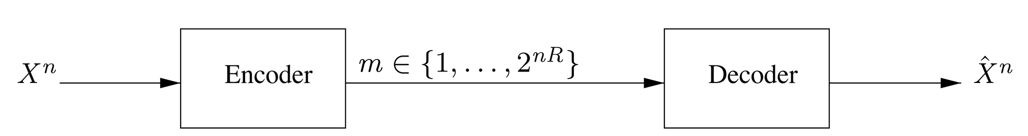
Rate-Distortion Coding



- $\{X_i\}_{i \in \mathbb{Z}}$: discrete-valued stochastic source
- To each coding scheme a distortion D is assigned as

$$D = \mathrm{E}\, d_n(X^n, \hat{X}^n) \triangleq \frac{1}{n}\sum_{i=1}^{n} Ed(X_i, \hat{X}_i)$$

- The goal is to find a coding scheme that for each given $\alpha > 0$ minimizes the Lagrangian cost $R + \alpha D$

Universal lossy compression

- The performance of each coding scheme depends on the source distribution.
- Does there exist a universal coding scheme that works well for any stationary ergodic source?
- Yes.
- Are there practical universal lossy compression algorithms?
- For a long time it was believed that no such algorithm exists.

Empirical distribution

- Consider binary alphabet $\mathcal{X} = \hat{\mathcal{X}} = \{0, 1\}$
- Let $\mathbf{m}(y^n)$ denote a 2×2^k matrix defining the $(k+1)^{\text{th}}$ order empirical distribution of y^n
- Example: $k = 2$,

$$\mathbf{m}(y^n) = \begin{bmatrix} m_{0,00} & m_{0,01} & m_{0,10} & m_{0,11} \\ m_{1,00} & m_{1,01} & m_{1,10} & m_{1,11} \end{bmatrix}$$

For $b \in \{0, 1\}$

$$m_{b,00}(y^n) = \frac{1}{n}\left|\{i : y_i = b, y_{i-2}^{i-1} = [0, 0]\}\right|$$

Conditional empirical entropy

- Conditional empirical entropy of y^n is defined as

$H_k(y^n) = H(Y_{k+1}|Y^k)$ when Y^{k+1} is distributed according to $\mathbf{m}$

- In the previous example:

$$H_2(y^n) = \sum_{b_1, b_2}(m_{0,[b_1,b_2]} + m_{1,[b_1,b_2]})\mathcal{H}(m_{0,[b_1,b_2]}, m_{1,[b_1,b_2]})$$

- Let $\alpha_1 = m_{0,[b_1,b_2]}$ and $\alpha_2 = m_{1,[b_1,b_2]}$. Then

$$\mathcal{H}(\alpha_1, \alpha_2) = H(X),$$

where $X \sim \mathrm{Bern}(\frac{\alpha_1}{\alpha_1 + \alpha_2})$

Universal lossy compression algorithm

An Exhaustive Search Algorithm:

For coding sequence x^n, find sequence $\hat{x}^n$ such that

$$\hat{x}^n = \arg\min_{y^n}\ [H_k(y^n) + \alpha d_n(x^n, y^n)],$$

and describe it to the decoder using Lempel-Ziv algorithm.

Theorem:
For any stationary ergodic source $\mathbf{X}$ and $k = o(\log n)$,

$$\mathrm{E}\left[\ell_{\mathrm{LZ}}(\hat{X}^n) + \alpha d(X^n, \hat{X}^n)\right] \to \min_{D}[R(\mathbf{X}, D) + \alpha D]$$

Exponential Complexity

Compression via Simulated Annealing

- To each sequence y^n assign energy $\mathcal{E}(y^n) = H_k(y^n) + \alpha d_n(x^n, y^n)$
- For $T > 0$, define a distribution on the space of all possible reconstructions as $p_T(y^n) \propto e^{-\mathcal{E}(y^n)/T}$
- Simulated annealing: gradually decreasing temperature while sampling from the distribution; finds $\hat{x}^n$ such that

$$\mathcal{E}(\hat{x}^n) \approx \min_{y^n} \mathcal{E}(y^n)$$

Theorem:[JW08]
Let $\mathbf{X}$ be a stationary and ergodic source. Then

$$\lim_{n\to\infty}\lim_{r\to\infty}\mathrm{E}\left[\frac{1}{n}\ell_{\mathrm{LZ}}\left(\hat{X}^n_{\alpha,r}(X^n)\right) + \alpha d_n(X^n, \hat{X}^n)\right] = \min_{D\geq 0}[R(\mathbf{X}, D) + \alpha D].$$

No bound on the required number of iterations

Another approach

- Let

$$(\mathrm{P1}):\quad \min_{y^n}[H_k(y^n) + \alpha d_n(x^n, y^n)]$$

$$(\mathrm{P2}):\quad \min_{y^n}\Big[\sum_{b_{k+1}}\sum_{\mathbf{b}}\lambda_{b_{k+1},\mathbf{b}} m_{b_{k+1},\mathbf{b}}(y^n) + \alpha d_n(x^n, y^n)\Big]$$

- Assume that we could find a set of coefficients $\{\lambda_{b_{k+1},\mathbf{b}}\}_{b_{k+1},\mathbf{b}}$ such that (P1) and (P2) have the same minimizers
- Why are we interested in this new representation?
- Note:

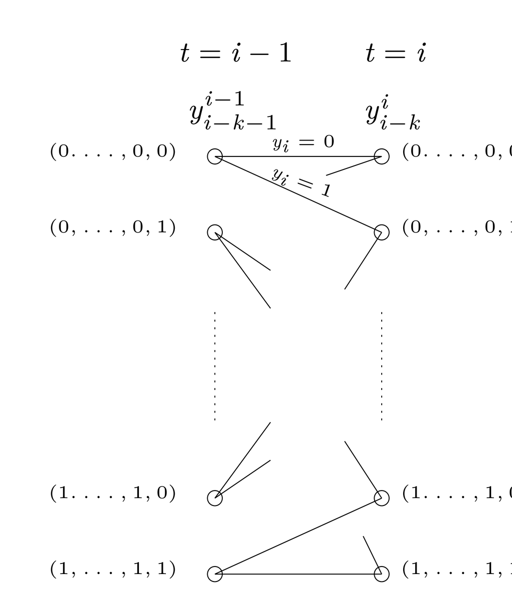
$$\sum_{b_{k+1},\mathbf{b}}\left[\lambda_{b_{k+1},\mathbf{b}} m_{b_{k+1},\mathbf{b}}(y^n) + \alpha d_n(x^n, y^n)\right]$$
$$= \frac{1}{n}\sum_{b_{k+1},\mathbf{b}}\left[\lambda_{b_{k+1},\mathbf{b}}\sum_{i=1}^{n}\mathbf{1}_{y_{i-k}^i = [\mathbf{b}, b_{k+1}]} + \alpha d(x_i, y_i)\right]$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left[\lambda_{y_i, y_{i-k}^{i-1}} + \alpha d(x_i, y_i)\right].$$

- (P2) can be solved using Viterbi algorithm

Viterbi algorithm

- Let the state at time i be $\mathbf{s}_i = (y_{i-k}^{i-1}, y_i)$
- One-to-one correspondence between the paths inside the trellis, sequence of states $\{s_i\}_{i=1}^n$, and also sequences y^n
- We want to find the path in the trellis minimizing $\sum_{i=1}^n w_i(y_{i-k}^i)$, where

$$w_i(\mathbf{s}_i) = \lambda_{y_i, y_{i-k}^{i-1}} + \alpha d(x_i, y_i)$$



Do such coefficients exist?

- Yes

Lemma

(P1) and (P2) have the same minimum values, if the coefficients are chosen according to

$$\lambda_{b_{k+1},\mathbf{b}} = \frac{\partial}{\partial m_{b_{k+1},\mathbf{b}}} H(\mathbf{m})\,\Big|_{\mathbf{m}_n^*}$$

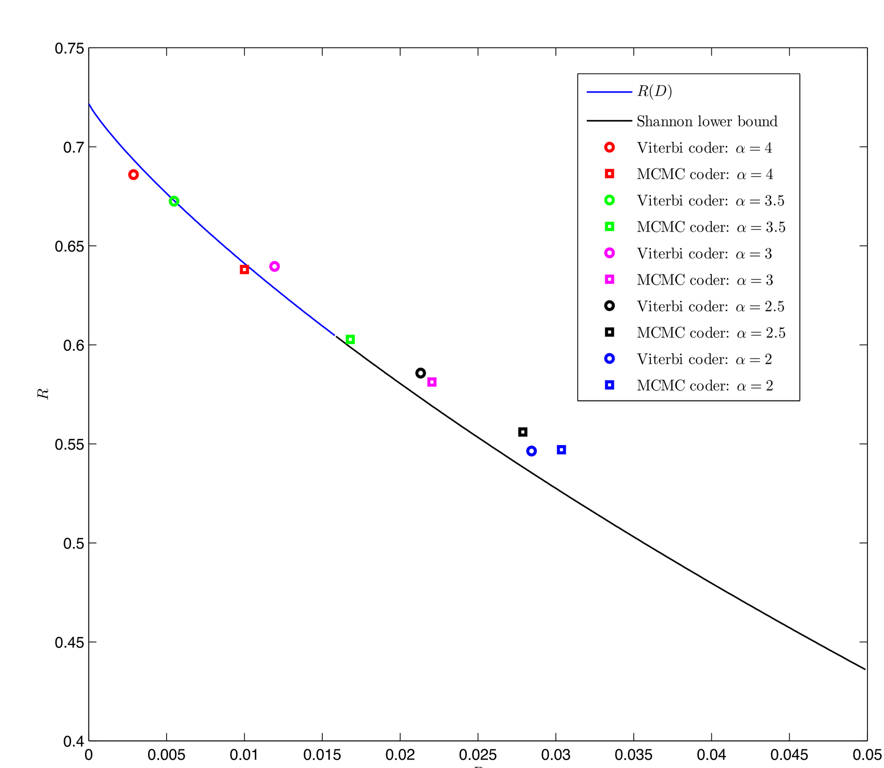
where $\mathbf{m}_n^* = \mathbf{m}_n^*(\hat{x}^n)$, where $\hat{x}^n$ is a minimizer of (P1)

- Do we have access to $\hat{x}^n$? Of course not!
- But, we can find an approximate version of $\mathbf{m}_n^*$ by solving a non-convex optimization problem.

Practical implementation of the new algorithm

- We need to approximate the coefficients
- Solving the non-convex optimization problem is hard
- Try simpler approximations for $\mathbf{m}^*$, like $\mathbf{m}(x^n)$
  - works well for small distortions
- For larger distortions, we can use iterative approach

Simulation results



Simulation results