Reduction of Decentralized Control Problems to Tractable Representations

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Abstract

We address the question of which decentralized control problems are tractable. The largest known class is the quadratically invariant systems. We broaden this set by characterizing a class of systems which are not quadratically invariant, but reduce to systems that are. We call this *internal quadratic invariance.* We give an algorithm for performing this reduction, and illustrate our method with examples.

Preliminaries

Matrices of Rational Functions

- Define $\mathcal{R}^{m\times n}$ to be the set of $m \times n$ matrices of rational functions. Upper-case variables generally belong to this set, unless otherwise specified.
- The ***normal rank*** of a matrix $A \in \mathcal{R}^{m\times n}$, denoted $\operatorname{nrank} A$, is the maximum value of $\operatorname{rank} A(s)$ over all $s \in \mathbb{C}$ at which A is defined. Note: $\operatorname{rank} A(s) \neq \operatorname{nrank} A$ only for finitely many values of s.
- If A is square and its determinant $\det A(s)$ is not zero for all s, then the inverse A^{-1} is a well-defined matrix of rational functions. Regardless of whether A has full normal rank or not, we can factor it as the product of two matrices with full normal rank:

Lemma 1. *Suppose $A \in \mathcal{R}^{m\times n}$. There exists a factorization $A = U_1U_2$, where $U_1 \in \mathcal{R}^{m\times r}$ is tall and full normal rank, $U_2 \in \mathcal{R}^{r\times n}$ is wide and full normal rank, and $r = \operatorname{nrank} A$.*

Closed-Loop Interconnections

We will routinely use the following vector signals:

$z \in \mathcal{R}^{m_1}$ (regulated outputs) $\quad w \in \mathcal{R}^{n_1}$ (exogenous inputs)
$y \in \mathcal{R}^{m_2}$ (sensed outputs) $\quad u \in \mathcal{R}^{n_2}$ (actuator inputs).

We write $m = m_1 + m_2$ and $n = n_1 + n_2$. The plant $P \in \mathcal{R}^{m\times n}$ maps inputs to outputs, and the controller $K \in \mathcal{R}^{n_2\times m_2}$ maps sensed outputs to actuator inputs:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad u = Ky$$

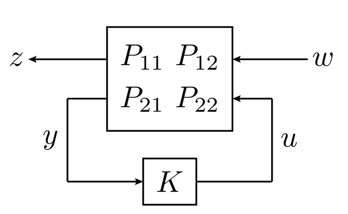


FIGURE 1: Closed-loop interconnection between a plant P and controller K. Note that the entries P_{ij} are matrices of transfer functions

- When we connect K to P as shown in Figure 1, the interconnection is ***well-posed*** if $(I - P_{22}K)$ is invertible.
- We consider sets of interconnections parametrized by $K \in S$. We call the subspace $S \subset \mathcal{R}^{n_2\times m_2}$ the ***information constraint***.
- We refer to the pair (P, S) when we want to consider all the well-posed interconnections between P and a controller belonging to S.

Let $f(P, K)$ denote the ***closed-loop map***: the matrix of rational functions that maps w to z when we interconnect P and K as shown in Figure 1:

$$z = f(P, K)w, \qquad \text{where: } f(P, K) = P_{11} + P_{12}K\,(I - P_{22}K)^{-1}P_{21}$$

More generally, when K satisfies an information constraint: $K \in S$, define the set of all possible well-posed closed-loop maps:

$$f(P, S) = \big\{ f(P, K) \,\big|\, K \in S,\ (I - P_{22}K) \text{ is invertible} \big\}$$

Quadratic Invariance

- We say that (P, S) is ***quadratically invariant*** (QI), if:

$$\text{for all } K \in S, \text{ we have } KP_{22}K \in S.$$

- Under additional technical conditions, quadratic invariance is necessary and sufficient for the information constraint S to be preserved under feedback.
- Constrained controller synthesis problems which are QI can be cast as a convex optimization problem This result is from [2]:

Theorem 2. *Suppose $P_{22} \in \mathcal{R}_{sp}^{m_2\times n_2}$, $S \subset \mathcal{R}_p^{n_2\times m_2}$ is a frequency aligned subspace, and (P, S) is QI. Then K is optimal for the problem*

$$\begin{aligned} &\textit{minimize} && \|f(P, K)\| \\ &\textit{subject to} && K \in S \end{aligned} \tag{1}$$

if and only if $K = -Q(I - P_{22}Q)^{-1}$ and Q is optimal for

$$\begin{aligned} &\textit{minimize} && \|P_{11} - P_{12}QP_{21}\| \\ &\textit{subject to} && Q \in S \end{aligned} \tag{2}$$

- Equation (2) is a convex optimization problem and can be easily solved in most cases.
- The QI systems are the broadest known class of tractable decentralized control problems.

Internal Quadratic Invariance

- We say that (P, S) and $(\tilde{P}, \tilde{S})$ are ***equivalent*** if they have the same sets of possible closed-loop maps *i.e.* $f(P, S) = f(\tilde{P}, \tilde{S})$.
- If (P, S) is not QI, but is equivalent to $(\tilde{P}, \tilde{S})$, which is QI, then we can apply Theorem 2 to (P', S'), and solve the resulting convex optimization problem.
- In this case, we say that (P, S) is ***internally quadratically invariant***.

Example

Consider the following pair, where P_{11}, B_i, C_i, G_i, and K_i are matrices of rational functions of compatible size.

$$\tilde{P} = \left[\begin{array}{c|ccc} P_{11} & 2B_1 & 3B_2 & 5B_2 \\ \hline \frac{1}{2}C_1 & G_1 & 0 & 0 \\ \frac{1}{3}C_1 & \frac{2}{3}G_1 & 0 & 0 \\ \frac{1}{5}C_2 & \frac{2}{5}G_2 & \frac{3}{5}G_3 & G_3 \end{array}\right] \qquad \tilde{S} = \left\{ \begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} \,\middle|\, K_i \in \mathcal{R} \right\}$$

The information constraint $\tilde{S}$ is the subspace of controllers with a block-diagonal structure. Now consider the pair:

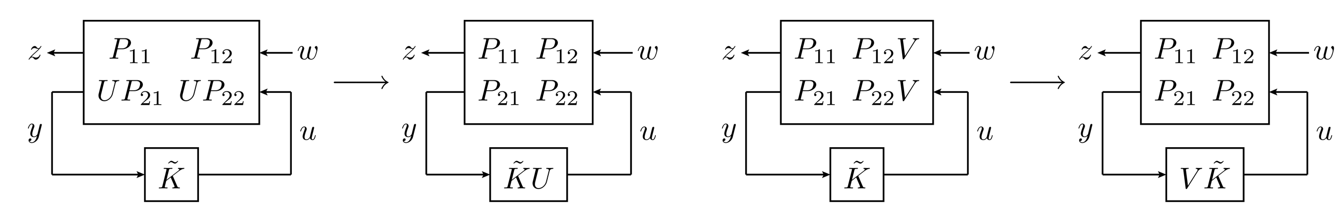
$$P = \left[\begin{array}{c|cc} P_{11} & B_1 & B_2 \\ \hline C_1 & G_1 & 0 \\ C_2 & G_2 & G_3 \end{array}\right] \qquad S = \left\{ \begin{bmatrix} K_1 & 0 \\ K_2 & K_3 \end{bmatrix} \,\middle|\, K_i \in \mathcal{R} \right\}.$$

The information constraint S is now the subspace of block lower-triangular controllers.

- $(\tilde{P}, \tilde{S})$ and (P, S) are equivalent. *i.e.* $f(\tilde{P}, \tilde{S}) = f(P, S)$.
- $(\tilde{P}, \tilde{S})$ is internally QI, since $(\tilde{P}, \tilde{S})$ is not QI, but (P, S) is.

Main Results

Consider the two transformations shown below:



$$\begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ and } S = \tilde{S}U \qquad \begin{bmatrix} \tilde{P}_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \text{ and } S = V\tilde{S}$$

FIGURE 2: Equivalent interconnections obtained by an ***output transformation*** (on the left), or an ***input transformation*** (on the right). In both cases, $(\tilde{P}, \tilde{S}) \to (P, S)$

Theorem 3. *Suppose (P, S) is an output (input) transformation of $(\tilde{P}, \tilde{S})$ under U.*

1. *The systems are equivalent: $f(\tilde{P}, \tilde{S}) = f(P, S)$*
2. *The transformation preserves quadratic invariance: $(\tilde{P}, \tilde{S})$ is QI $\implies$ (P, S) is QI.*
3. *If U is wide (tall) and has full normal rank, then the converse holds: if (P, S) is QI, then $(\tilde{P}, \tilde{S})$ is QI.*

We say (P, S) is ***minimal*** if $\begin{bmatrix} P_{21} & P_{22} \end{bmatrix}$ and $\begin{bmatrix} P_{12}^T & P_{22}^T \end{bmatrix}^T$ are wide and have full normal rank.

Theorem 4. *For any pair (P, S),*

1. *There exists an output/input transformation that makes (P, S) minimal.*
2. *A system is internally QI if its minimal form is QI.*

In other words, we need only compute the minimal form of a system and test whether it is QI. If it is not, there does *not* exist any input or output transformations that will yield a QI system.

Example: Networked System

In the example below, we see that whether a system is QI or not can depend on how we group the blocks, *i.e.* what we call "P" and "K". The method of Theorem 4 allows us to disambiguate such cases.

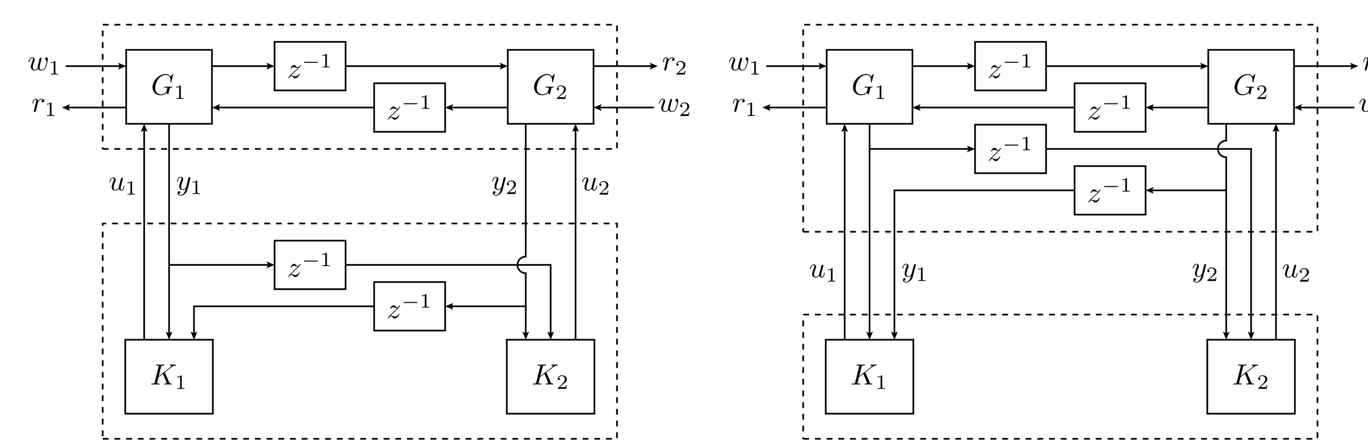


FIGURE 3: A system consisting of two coupled sub-systems with controllers that receive delayed measurements, represented in two different ways. The version on the left is quadratically invariant, while the version on the right is not.

References

[1] T. Kailath. *Linear systems.* Prentice-Hall Englewood Cliffs, NJ, 1980.

[2] M. Rotkowitz and S. Lall. Decentralized control information structures preserved under feedback. In *IEEE Conference on Decision and Control*, volume 1, pages 569–575, 2002.

[3] M. Rotkowitz and S. Lall. A characterization of convex problems in decentralized control. *IEEE Transactions on Automatic Control*, 51(2):274–286, 2006.