Bounds and Policies for Multi-Period Investment

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Problem setup

- manage portfolio of n assets for time $t = 0, \dots, T$
  - $x_t \in \mathbf{R}^n$ portfolio positions at time t
  - $u_t \in \mathbf{R}^n$ trades at time t
- portfolio propagates according to

$$x_{t+1} = \mathbf{diag}(r_{t+1})(x_t + u_t), \quad t = 0, \dots, T-1$$

- $r_t \in \mathbf{R}^n$ returns, first and second moments known:

$$\mathbf{E}(r_t) = \bar{r}_t, \quad \mathbf{E}(r_t - \bar{r}_t)(r_t - \bar{r}_t)^T = \Sigma_t, \quad t = 1, \dots, T$$

- denote the post-trade portfolio by $x_t^+ = (x_t + u_t)$

Goal

- find policies $\phi_0, \dots, \phi_T : \mathbf{R}^n \to \mathbf{R}^n$ to minimize $J = \mathbf{E} \sum_{t=0}^{T} \ell_t(x_t, u_t)$
- where we set $u_t = \phi_t(x_t)$, and

$$\ell_t(x,u) = \begin{cases} \mathbf{1}^T u + \psi_t(x,u) & x + u \in \mathcal{C}_t \\ \infty & \text{otherwise} \end{cases}$$

  - $\mathcal{C}_t$ portfolio constraints
  - $\mathbf{1}^T u_t$ gross cash invested
  - $\psi_t : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ transaction or position cost

Portfolio constraint examples

position limits	$x_t^{\min} \le x^+ \le x_t^{\max}$
proportional position limits	$x^+ \le (\mathbf{1}^T x^+)\alpha_t$
total value minimum	$\mathbf{1}^T x^+ \ge v_t^{\min}$
terminal portfolio constraint	$x_T^+ = x^{\text{term}}$
leverage limits	$\mathbf{1}^T (x^+)_- \le \eta_t \mathbf{1}^T x^+$
sector exposure limits	$s_t^{\min} \le F_t x^+ \le s_t^{\max}$
sector neutrality	$F_t x^+ = 0$
concentration limits	$\sum_{i=1}^{p} (x^+)_{[i]} \le \beta_t \mathbf{1}^T x^+$
variance risk limits	$(x^+)^T \Sigma_{t+1} x^+ \le \gamma_t$
homogeneous risk limits	$\lVert \Sigma_{t+1}^{1/2} x^+ \rVert_2 \le \delta_t \mathbf{1}^T x^+$

Transaction and position cost examples

broker commission	$(\kappa_t^{\text{buy}})^T u_+ + (\kappa_t^{\text{sell}})^T u_-$
bid-ask spread	$\kappa_t^T \lvert u \rvert$
quadratic price impact	$s_t^T u^2$
3/2 power price impact	$s_t^T \lvert u \rvert^{3/2}$
borrowing/shorting fee	$c_t^T (x^+)_-$
quadratic risk penalty	$\lambda_t (x^+)^T \Sigma_t x^+$
std. dev. risk penalty	$\lambda_t \lVert \Sigma_t^{1/2} x^+ \rVert_2$

Dynamic programming 'solution'

- Bellman recursion: $V_{T+1} = 0$, for $t = T, T-1, \dots, 0$

$$V_t(x) = \inf_u \left(\ell_t(x,u) + \mathbf{E}\, V_{t+1}(\mathbf{diag}(r_{t+1})(x+u)) \right),$$

  abstractly $V_t = \mathcal{T}_t V_{t+1}, \quad t = T, T-1, \dots, 0$
- optimal policies

$$\phi_t^\star(x) \in \operatorname*{argmin}_u \left(\ell_t(x,u) + \mathbf{E}\, V_{t+1}(\mathbf{diag}(r_{t+1})(x+u)) \right)$$

- intractable to compute V_t in most cases (quadratic a special case)

Performance bounds

- if we have $V_t^{\text{lb}} \le V_t$, $t = 0, \dots, T$, then $V_t^{\text{lb}}(x_0) \le V_0(x_0) = J^\star$
- sufficient condition:

$$V_{T+1}^{\text{lb}} = V_{T+1} = 0, \quad V_t^{\text{lb}} \le \mathcal{T}_t V_{t+1}^{\text{lb}}, \quad t = T, T-1, \dots, 0$$

- if V_t^{lb} quadratic then maximizing bound is convex:

$$\begin{array}{ll} \text{maximize} & V_0^{\text{lb}}(x_0) \\ \text{subject to} & V_t^{\text{lb}} \le \mathcal{T}_t V_{t+1}^{\text{lb}}, \quad t = 0, \dots, T \\ & V_{T+1}^{\text{lb}} = 0 \end{array}$$

Suboptimal Policies

- **approximate dynamic programming**
  - replace V_t with approximation $\hat{V}_t$ (*e.g.*, $\hat{V}_t = V_t^{\text{lb}}$)
  - policy: $\hat{\phi}_t(x) \in \operatorname{argmin}_u \left(\ell_t(x,u) + \mathbf{E}\, \hat{V}_{t+1}(\mathbf{diag}(r_{t+1})(x+u)) \right)$
  - $\hat{V}$ quadratic then computing policy costs $\mathcal{O}(n^3)$ operations
- **model predictive control**
  - solve open loop problem, replace all random quantities with expected values
  - repeat at each iteration
  - costs $\mathcal{O}((T-t)n^3)$ operations to compute policy

Numerical examples

- $n = 30$ assets, horizon $T = 99$, $x_0 = x_T = 0$
- returns IID log-normal, *i.e.*, $\log(r_t) \sim \mathcal{N}(\mu, \tilde{\Sigma})$
- one pure quadratic example and four others:
  - quadratic: $\psi(x,u) = s^T u^2 + \lambda (x^+)^T \Sigma x^+$
  - other cases: $\psi(x,u) = c^T (x^+)_- + \kappa^T \lvert u \rvert + s^T u^2 + \lambda (x^+)^T \Sigma x^+$
  - consists of: shorting cost, bid-ask spread/broker costs, quadratic price impact, risk penalty
- constraint sets:
  - unconstained: $\mathcal{C}_t = \mathbf{R}^n$
  - long-only: $\mathcal{C}_t = \mathbf{R}_+^n$
  - leverage limit: $\mathcal{C}_t = \{x^+ \mid \mathbf{1}^T (x^+)_- \le \eta \mathbf{1}^T x^+\}$, $\eta = 0.3$
  - sector neutral: $\mathcal{C}_t = \{x^+ \mid F x^+ = 0\}$, $F \in \mathbf{R}^{2 \times n}$

Results

- ADP policy: 50000 Monte Carlo samples of 100 time-periods
- means solving 5 million QPs; takes about 3 minutes with 8 cores
- MPC policy: ran 5000 Monte Carlo samples, took several hours

Example	Lower bound	ADP performance	MPC performance
quadratic	-450.1	-450.3	-444.3
unconstrained	-132.6	-131.9	-130.6
long-only	-41.3	-41.0	-40.6
leverage limit	-87.5	-85.6	-84.7
sector neutral	-121.3	-118.9	-117.5

Leverage limit ex., ADP policy