Capacity Region of the Broadcast Channel with Two Deterministic Channel State Components

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Basics: Broadcast channel

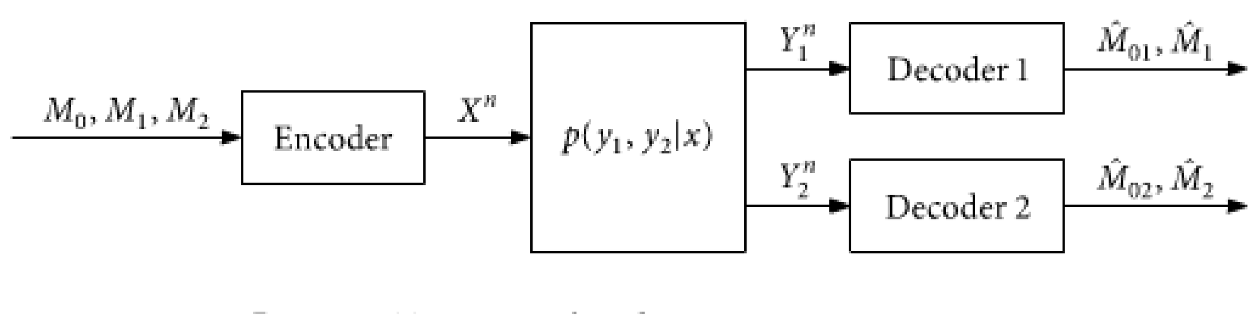

Figure: Two-receiver broadcast communication system [1]

A 2-receiver *discrete memoryless broadcast channel* (DM-BC) model $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_2 \times \mathcal{Y}_2)$ consists of three finite sets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$, and a collection of conditional pmfs $p(y_1, y_2|x)$ on $\mathcal{Y}_1 \times \mathcal{Y}_2$ (one for each input symbol x). We consider when $M_0 = \emptyset$.

A $(2^{nR_1}, 2^{nR_2})$ code for a DM-BC consists of

- two message sets $[1 : 2^{nR_1}], [1 : 2^{nR_2}]$,
- an encoder that assigns a codeword $x^n(m_1, m_2)$ to each message pair $(m_1, m_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]$, and
- two decoders, where decoder 1 assigns an estimate $\hat{m}_1 \in [1 : 2^{nR_1}]$ or an error message e to each received sequence y_1^n, and decoder 2 assigns an estimate $\hat{m}_2 \in [1 : 2^{nR_2}]$ or an error message e to each received sequence y_2^n.

We assume that the message pair (M_1, M_2) is uniformly distributed over $[1 : 2^{nR_1}], [1 : 2^{nR_2}]$. The average probability of error is defined as

$$P_e^n = \mathrm{P}\{\hat{M}_1 \neq M_1 \text{ or } \hat{M}_2 \neq M_2\}.$$

A rate pair (R_1, R_2) is said to be achievable for the DM-BC if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $\lim_{n\to\infty} P_e^n = 0$.

The capacity region $\mathscr{C}$ of the DM-BC is the closure of the set of achievable rate pairs (R_1, R_2).

Broadcast channel with two deterministic channel states

The 2-receiver *broadcast channel with two deterministic channel states* (or BC-TDCS in short) is a discrete memoryless broadcast channel with random state $(\mathcal{X} \times \mathcal{S}, p(s)p(y_1, y_2|x, s), \mathcal{Y}_1 \times \mathcal{Y}_2)$, where $S = (S_1, S_2) \in \{1, 2\}^2$, $p_{S_1}(1) = p_1, p_{S_1}(2) = 1 - p_1 = \bar{p}_1$ and $p_{S_2}(1) = p_2, p_{S_2}(2) = \bar{p}_2$, and the outputs

$$Y_1 = \begin{cases} f_1(X) & \text{if } S_1 = 1, \\ f_2(X) & \text{if } S_1 = 2, \end{cases}$$
$$Y_2 = \begin{cases} f_1(X) & \text{if } S_2 = 1, \\ f_2(X) & \text{if } S_2 = 2 \end{cases}$$

for some deterministic functions f_1 and f_2 of the input X.

References

A. El Gamal and Y. H. Kim, *Network Information Theory*, 1st ed. Cambridge University Press, 2011.

H. Kim, Y.-K. Chia, and A. El Gamal, "A note on broadcast channels with stale state information at the transmitter," *CoRR*, vol. abs/1309.7437, 2013.

Theorem: Capacity Region of the BC-TDCS

Without loss of generality, assume $p_1 \geq p_2$. The capacity region of the BC-TDCS $(\mathcal{X} \times \mathcal{S}, p(s)p(y_1, y_2|x, s), \mathcal{Y}_1 \times \mathcal{Y}_2)$ with the state known only at the receivers is the convex hull of the set of all rate pairs (R_1, R_2) such that

$$\begin{aligned} R_1 &\leq I(U_1; Y_1|S), \\ R_2 &\leq I(U_2; Y_2|S), \\ R_1 + R_2 &\leq I(U_1; Y_1|S) + I(U_2; Y_2|S) - I(U_1; U_2) \end{aligned} \quad (1)$$

for some $p(x)$ and either $(U_1, U_2) = (f_1, f_2)$, $(U_1, U_2) = (X, \emptyset)$, or $(U_1, U_2) = (\emptyset, X)$.

Example 1: Blackwell channel with state [2]

As an example of a BC-TDCS, consider the following. The functions f_1 and f_2 for this example are depicted in Figure 2.

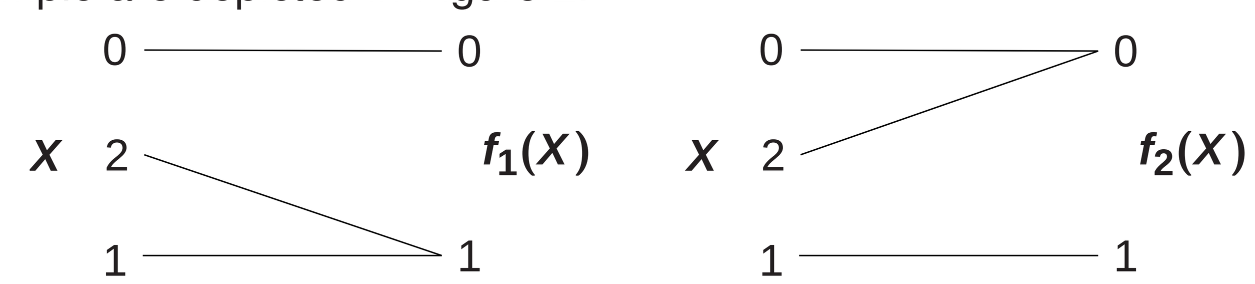

Figure: The deterministic components of the Blackwell channel with state.

The capacity region of the Blackwell channel with state known only to the receivers is the convex hull of the union of:

$$\mathscr{R}_3' = \{(R_1, R_2) : \; R_1 \leq H(\alpha_0) - \bar{p}_1 \bar{\alpha}_1 H(\alpha_0/\bar{\alpha}_1), \; R_2 \leq \bar{p}_2 \bar{\alpha}_0 H(\alpha_1/\bar{\alpha}_0) \text{ for some } \alpha_0, \alpha_1 \geq 0, \alpha_0 + \alpha_1 \leq 1\}, \text{ and}$$

$$\mathscr{R}_4' = \{(R_1, R_2) : \; R_1 \leq p_1 \bar{\alpha}_1 H(\alpha_0/\bar{\alpha}_1), \; R_2 \leq H(\alpha_1) - p_2 \bar{\alpha}_0 H(\alpha_1/\bar{\alpha}_0) \text{ for some } \alpha_0, \alpha_1 \geq 0, \alpha_0 + \alpha_1 \leq 1\}.$$

The capacity region with state for the following cases are plotted in Figure 3.

- For $(p_1, p_2) = (0.5, 0.5)$, the two channels are statistically identical, hence the capacity region coincides with the time-division region.
- For $(p_1, p_2) = (1, 0)$, the channel reduces to the Blackwell channel with no state.
- For (p_1, p_2) in between these two extreme cases, the capacity region is established by our theorem.

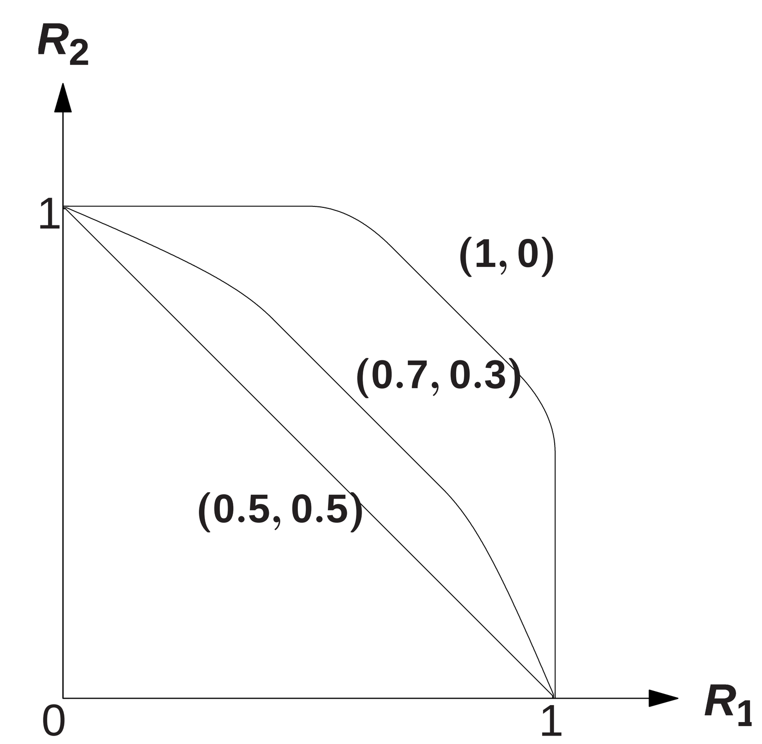

Figure: Capacity region of the Blackwell channel with the state.

Example 2: Finite-state BC-TDCS

Consider the BC-TDCS with the state known only at the receivers with $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T$:

$$Y_1 = \begin{cases} h_{11}X_1 + h_{12}X_2 & \text{if } S_1 = 1, \\ h_{21}X_1 + h_{22}X_2 & \text{if } S_1 = 2, \end{cases}$$
$$Y_2 = \begin{cases} h_{11}X_1 + h_{12}X_2 & \text{if } S_2 = 1, \\ h_{21}X_1 + h_{22}X_2 & \text{if } S_2 = 2, \end{cases} \quad (2)$$

where the channel matrix is full-rank, $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{X}_1 = \mathcal{X}_2 = [0 : K - 1]$, and the arithmetic is over the finite field.
The capacity region is

$$\mathscr{C} = \mathrm{co}\{(0, 0), (\log K, 0), (0, \log K), (p_1 \log K, \bar{p}_2 \log K)\}.$$

Figure 4 plots the capacity region for $(p_1, p_2) = (0.5, 0.5), (0.7, 0.4)$, and $(1, 0)$.

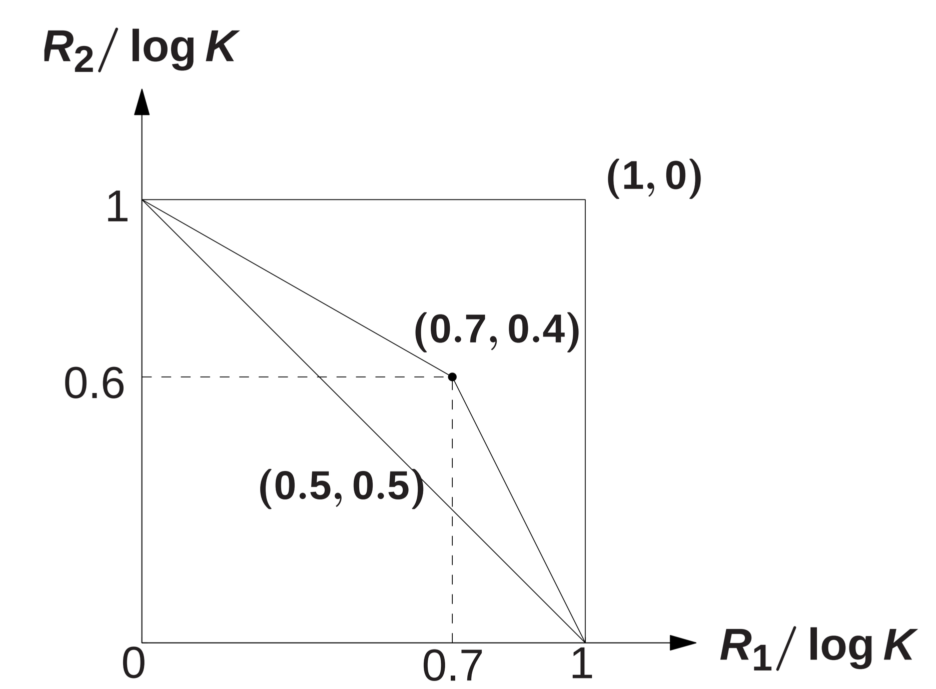

Figure: Capacity region of the Finite Field BC-TDCS.

Connection to wireless channels: The degree-of-freedom (DoF) of the corresponding broadcast channel is $p_1 + \bar{p}_2$.

Proof of the theorem

To obtain the capacity region, we need to prove that $(R_1, R_2) \in \mathscr{C}$ is achievable(achievability) and $(R_1, R_2) \notin \mathscr{C}$ is not achievable(converse).
The key of our proof is that UV upper bound coincides with Marton inner bound for BC-TDCS (but the two bounds do not match in general).

Summary of results

- We establish the capacity region of a new class of broadcast channels—our setting does not belong to any class of broadcast channels with previously known capacity region.
- We establish the capacity region of a nontrivial class of broadcast channels with state known at the receivers—a setting with very few known results.
- We provide yet another class of broadcast channels for which Marton coding is optimal.
- Our channel model can be used to approximate certain fading broadcast channels in high SNR.

Comments

This is joint work with Abbas El Gamal. The paper is accepted for ISIT 2014 and can be found at axXiv.