Gauss Quadrature via Linear Programming

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Introduction

Numerical Integration is one of the most important problems in scientific computing and for univariate integrals the optimal method, Gauss quadrature, has been long known.
Unfortunately, traditional Gauss quadrature does not generalize to the multivariate setting and the optimal method for multivariate integration has been an open problem.
In this poster, we present a reformulation of Gauss quadrature via linear programming (and not the traditional formulation via orthogonal polynomials). Then the reformulation naturally generalizes to multivariate setting.

Background

Gauss Quadrature

A quadrature is an approximation rule to integration of the following form

$$\int f d\mu \approx \sum_{i=1}^{k} w_i f(x_i),$$

i.e.$\int f d\mu$ is approximated with k point evaluations of $f(x)$. Roughly speaking, a good quadrature achieves high accuracy with small k.

One figure of merit that quantifies the objective "high accuracy" is the degree of polynomials up to which the quadrature is exact. The celebrated *Gauss quadrature* is exact on polynomials of degree less than $2k$, which is provably optimal for 1 dimensional integrals.

Unfortunately, Gauss quadrature does not generalize to multidimensional integrals.

Theory

Reformulation of Gauss Quadrature

Traditionally, the theory of Gauss quadrature is developed with polynomial division and the fundamental theorem of algebra. Unfortunately such results do not have an appropriate generalization to the multivariate setting.

The following theorem recasts Gauss quadrature as a solution to the optimization problem.

Theorem The solution to the following infinite dimensional linear program is Gauss quadrature

$$\begin{array}{ll} \text{minimize} & \int x^n \, d\mu \\ \text{subject to} & \int x^i \, d\mu = \int x^i \, dq \quad \text{for } i = 1, \cdots, n, \\ & \mu \geq 0 \end{array}$$

Where μ, a Borel measure, is the optimization variable.

Generalization to Multidimensions

The above reformulation of Gauss quadrature has a natural generalization to the multidimensional setting.

Theorem There is a solution to the following infinite dimensional linear program has support at most n

$$\begin{array}{ll} \text{minimize} & \int r \, d\mu \\ \text{subject to} & \int p^{(i)} \, d\mu = \int p^{(i)} \, dq \quad \text{for } i = 1, \cdots, n, \\ & \mu \geq 0 \end{array}$$

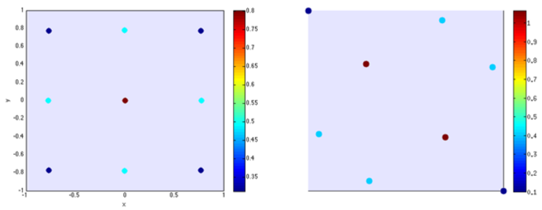
Where μ, a Borel measure, is the optimization variable.

In practice, the solution μ^* often has support less than n.

Examples

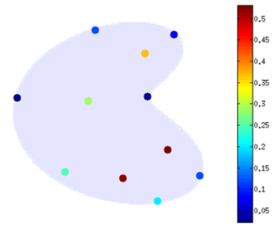
2D Unit Square

The left is standard gauss quadrature while the right is quadrature obtained via linear programming. Both have degree 6 but the right one outperforms the left one by using fewer nodes.



Arbitrary Domain

Standard Gauss quadrature simply fails to generalize to arbitrary domains without symmetry. The figure below is a quadrature of degree 6 obtained via linear programming.



Surface Integration

This method can also be used to produce quadratures for integration on manifolds. The figure below is a quadrature of degree 6 obtained via linear programming.