Minimum Time Speed Optimization Along a Fixed Path

Thomas Lipp and Stephen Boyd
Information Systems Laboratory, Electrical Engineering Department, Stanford University

Motivation
- minimum time trajectories are desired in a wide range of applications:
  - surveillance
  - flight paths
  - machine paths
  - racing
- trajectory generation problems are hard
- currently, trajectory generation is often divided into two parts
  - 1) a feasible path is generated through obstacles
  - 2) the time to traverse that path is optimized
- this research focuses on the second step of trajectory generation
- already studied for robotic manipulators; can be generalized to other vehicles

Dynamics
- vehicles have a generalized position \( q \in \mathbb{R}^p \)
- vehicles take control inputs \( u \in \mathbb{R}^r \)
- vehicles have dynamics:
  \[
  R(q)u = M(q)\dot{q} + C(q, \dot{q})q + d(q)
  \]
- \( R : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times r} \): controls matrix
- \( M : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times m} \): mass matrix
- \( S_{p \times m}^+ \): set of positive definite \( p \times m \) matrices
- \( C : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times r} \): centrifugal matrix, linear in \( \dot{q} \)
- \( d : \mathbb{R}^p \rightarrow \mathbb{R}^m \): position dependent force

Control Constraints
- vehicles have control constraints:
  \[
  (q^2, \dot{q}, u) \in \mathbb{C}(q)
  \]
- \( \mathbb{C}(q) \subseteq \mathbb{R}^{2p \times 2m} \): a set valued mapping to convex sets
- control limits can be dependent on generalized position
- path constraints (i.e. speed limits) can also be represented

Path
- vehicles travel a path:
  \[
  s : [0, 1] \rightarrow \mathbb{R}^p
  \]
- \( s(\theta(t)) = q(t), \quad t \in [0, T] \)
- \( \theta : [0, T] \rightarrow [0, 1] \)
- \( \theta(0) = 0, \quad \theta(T) = 1, \quad \dot{\theta} > 0 \)

Problem Statement

minimize \( T \)
subject to \( R(q(t))u(t) = M(q(t))\dot{q}(t) + C(q(t), \dot{q}(t))q(t) + d(q(t)), \quad t \in [0, T] \)
\[
\begin{align*}
\mathbf{s}(\theta(t)) &= q(t), & & t \in [0, T] \\
\mathbf{s}(\theta(T)) &= q(T) \\
\mathbf{c}(q(t)) &= \mathbb{C}(q), & & t \in [0, T]
\end{align*}
\]
- the functions \( s \) and \( u \) are optimization parameters
- \( R, M, C, d, s, \) and \( C \) are problem data
- through a clever change in variables, known since the 1970’s, this problem becomes a convex optimization problem
- convexity guarantees global optimality of solution
- discretize \( \theta \) to solve the problem
- the discrete convex optimization problem can be solved very efficiently
- time to solve is linear in the discretization of \( \theta \)

Example: Spacecraft
- a spacecraft modeled as a point mass with a force that can be applied in any direction subjected to gravity
  \[
  q = (x, y, z), \quad u = (f_x, f_y, f_z)
  \]
- \( M = mI, \quad C = 0, \quad d = -gme_3, \quad R = I \)
- \( m \): mass
- \( g \): acceleration due to gravity
- \( e_3 \): unit vector in the \( z \) direction
- \( \mathbb{C} = \{ (q^2, \dot{q}, u) | \|u\| \leq u_{\text{max}} \} \)
- \( u_{\text{max}} \): maximum force that the thruster can produce

Example: Car
- a car modeled using the friction circle assumption and treated as front wheel drive (less force available when accelerating)
  \[
  q = (x, y), \quad u = (f_{\text{avg}}, f_{\text{lat}})
  \]
- \( M = mI, \quad C = 0, \quad d = 0, \quad R = \begin{bmatrix} \cos(\phi(q)) & -\sin(\phi(q)) \\ \sin(\phi(q)) & \cos(\phi(q)) \end{bmatrix} \)
- \( \phi : R \rightarrow \mathbb{R} \): orientation of the vehicle
- \( \mathbb{C} = \{ (q^2, \dot{q}, u) | \|u\| \leq u_{\text{max}} F_N, \quad |f_{\text{avg}}| \leq W_f u_{\text{max}} F_N \} \)
- \( u_{\text{max}} \): friction between the road and wheels
- \( W_f \): percentage of mass supported by the front tires
- \( F_N = mg \): normal force to the road

Results

- more complex models fit in this formulation
- vectored thrust, variable friction, banked turns, aerodynamic drag
- can be applied to planes, underwater vehicles, ducted fans

Example: Extensions
- controls track the desired path
- computation took 16.8 milliseconds
- could be implemented with feedback or recomputation to reduce error

Figure 1: resultant path from applying minimum time control to a car example with \( \theta \) discretized at 100 points

Figure 2: forces applied

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