Minimum Time Speed Optimization Along a Fixed Path

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Motivation

- minimum time trajectories are desired in a wide range of applications:
  - surveillance
  - flight paths
  - machine paths
  - racing
- trajectory generation problems are hard
- currently, trajectory generation is often divided into two parts
  - 1) a feasible path is generated through obstacles
  - 2) the time to traverse that path is optimized
- this research focuses on the second step of trajectory generation
- already studied for robotic manipulators; can be generalized to other vehicles

Dynamics

- vehicles have a generalized position $q \in \mathbf{R}^p$
- vehicles take control inputs $u \in \mathbf{R}^r$
- vehicles have dynamics:

$$R(q)u = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + d(q)$$

  - $R : \mathbf{R}^p \to \mathbf{R}^{p \times r}$: controls matrix
  - $M : \mathbf{R}^p \to \mathbf{S}^p_{++}$: mass matrix
  - $\mathbf{S}^p_{++}$: set of positive definite $p \times p$ matrices
  - $C : \mathbf{R}^{2p} \to \mathbf{R}^{p \times p}$: centrifugal matrix, linear in $\dot{q}$
  - $d : \mathbf{R}^p \to \mathbf{R}^p$: position dependent force

Control Constraints

- vehicles have control constraints:

$$(\dot{q}^2, \ddot{q}, u) \in \mathcal{C}(q)$$

  - $\mathcal{C}(q) \subseteq \mathbf{R}^{p \times p \times m}$: a set valued mapping to convex sets
- control limits can be dependent on generalized position
- path constraints (i.e. speed limits) can also be represented

Path

- vehicles travel a path:

$$s : [0,1] \to \mathbf{R}^p$$

  - $s(\theta(t)) = q(t), \quad t \in [0,T]$
  - $\theta : [0,T] \to [0,1]$
  - $\theta(0) = 0, \quad \theta(T) = 1, \quad \dot{\theta} > 0$

Problem Statement

$$\begin{array}{ll} \text{minimize} & T \\ \text{subject to} & R(q(t))u(t) = M(q(t))\ddot{q}(t) + C(q(t),\dot{q}(t))\dot{q}(t) + d(q(t)), \quad t \in [0,T] \\ & s(\theta(t)) = q(t), \quad t \in [0,T] \\ & (\dot{q}(t)^2, \ddot{q}(t), u(t)) \in \mathcal{C}(q(t)), \quad t \in [0,T] \end{array}$$

- the functions θ and u are optimization parameters
- R, M, C, d, s, and $\mathcal{C}$ are problem data
- through a clever change in variables, known since the 1970's, this problem becomes a convex optimization problem
- convexity guarantees global optimality of solution
- discretize θ to solve the problem
- the discrete convex optimization problem can be solved very efficiently
- time to solve is linear in the discretization of θ

Example: Spacecraft

- a spacecraft modeled as a point mass with a force that can be applied in any direction subjected to gravity
- $q = (x, y, z), \quad u = (f_x, f_y, f_z)$
-

$$M = mI, \quad C = 0, \quad d = -gme_3, \quad R = I$$

  - m: mass
  - g: acceleration due to gravity
  - e_3: unit vector in the z direction
-

$$\mathcal{C} = \left\{ (\dot{q}^2, \ddot{q}, u) \mid \|u\|_2 \le u_{\max} \right\}$$

  - $u_{\max}$: maximum force that the thruster can produce

Example: Car

- a car modeled using the friction circle assumption and treated as front wheel drive (less force available when accelerating)
- $q = (x, y), \quad u = (f_{\text{long}}, f_{\text{lat}})$
-

$$M = mI, \quad C = 0, \quad d = 0, \quad R = \begin{bmatrix} \cos(\phi(q)) & -\sin(\phi(q)) \\ \sin(\phi(q)) & \cos(\phi(q)) \end{bmatrix}$$

  - $\phi : \mathbf{R} \to \mathbf{R}$: orientation of the vehicle
-

$$\mathcal{C} = \left\{ (\dot{q}^2, \ddot{q}, u) \mid \|u\|_2 \le \mu_s F_N, \quad |f_{\text{long}}| \le W_f \mu_s F_N \right\}$$

  - μ_s: friction between the road and wheels
  - W_f: percentage of mass supported by the front tires
  - $F_N = mg$: normal force to the road

Example: Extensions

- more complex models fit in this formulation
- vectored thrust, variable friciton, banked turns, aerodynamic drag
- can be applied to planes, underwater vehicles, ducted fans

Results

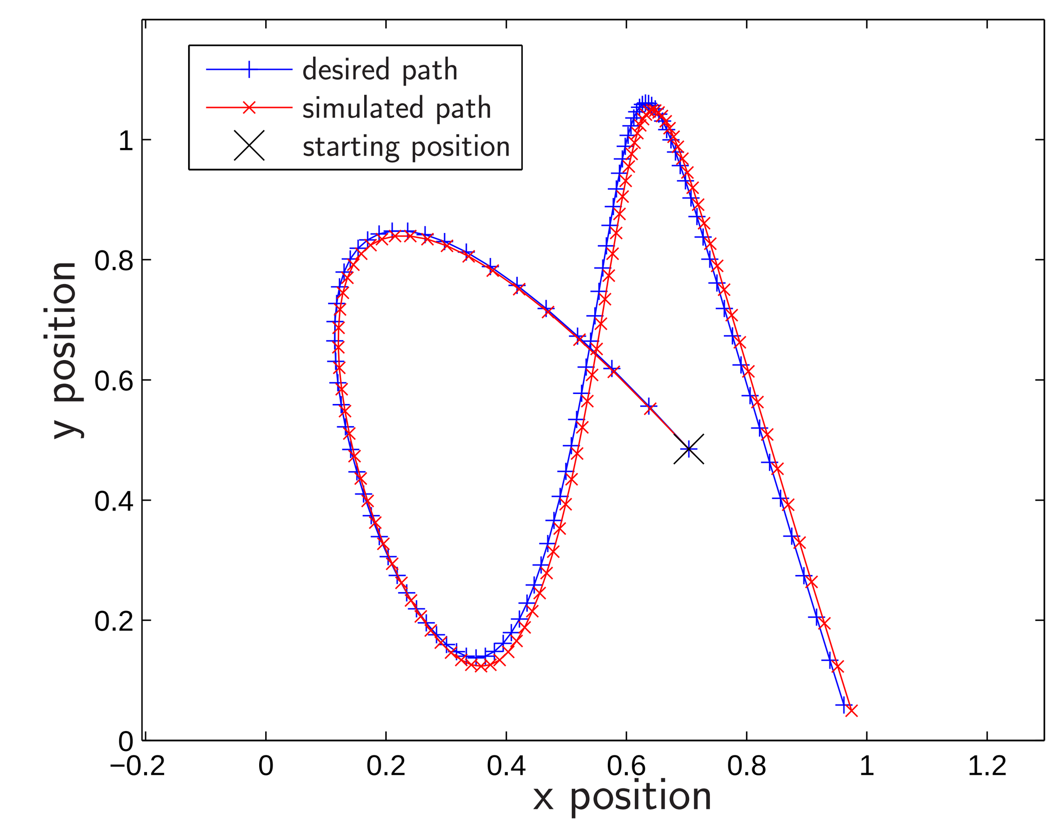

Figure 1: resultant path from applying minimum time control to a car example with θ discretized at 100 points

- controls track the desired path
- computation took 16.8 milliseconds
- could be implemented with feedback or recomputation to reduce error

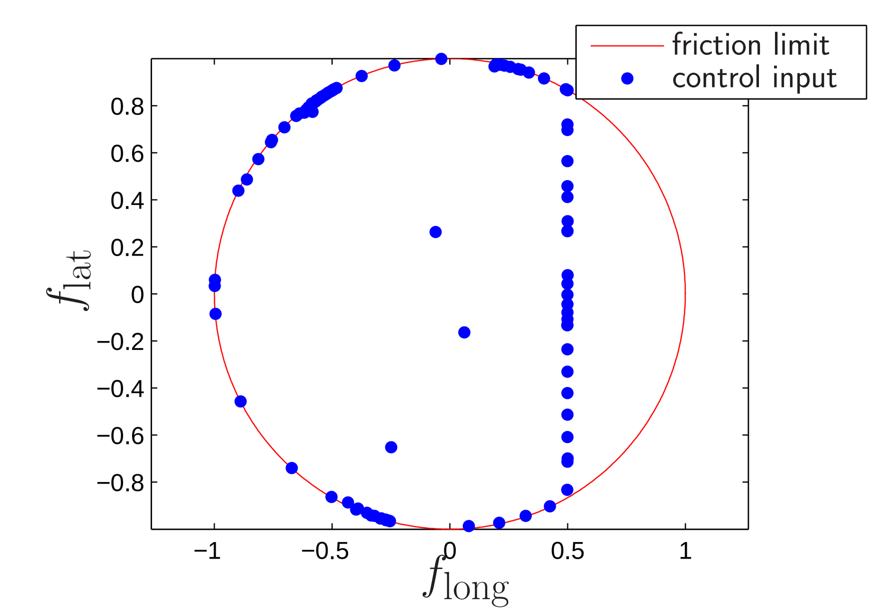

Figure 2: forces applied

- forces obey the force constraints
- forces are at their limits as expected, otherwise faster travel is possible

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