Sigmoidal programming

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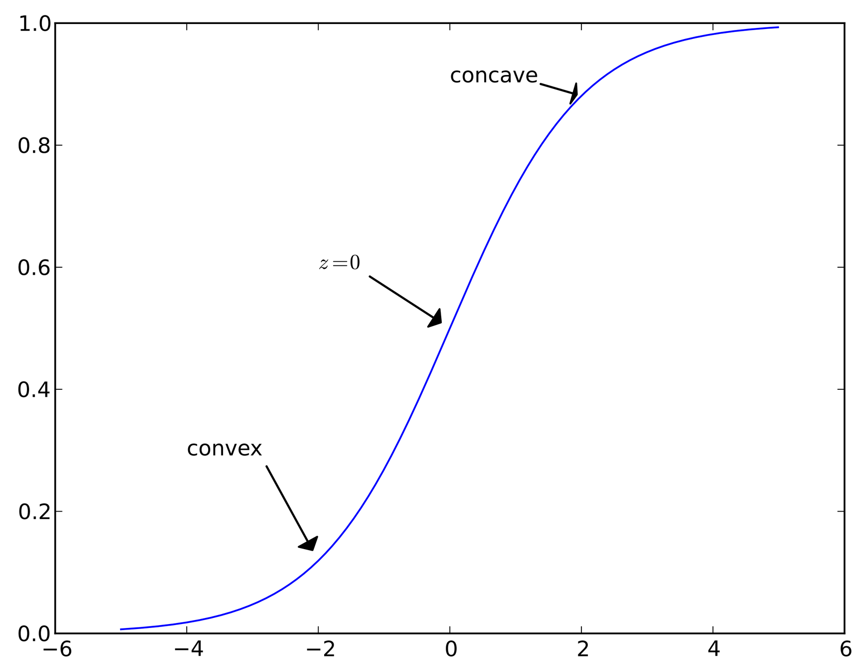
Sigmoidal functions

A continuous function $f : [l, u] \to \mathbf{R}$ is called *sigmoidal* if it is either convex, concave, or convex for $x \le z \in \mathbf{R}$ and concave for $x \ge z$.



Examples of sigmoidal functions:

- Logistic function: $\mathrm{logistic}(x) = 1/(1 + \exp(x))$.
- Normal CDF
- All linear, convex, or concave functions
- Step functions

Problem statement

Define the *sigmoidal programming* problem:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n f_i(x_i) \\ \text{subject to} & x \in \mathcal{C}. \end{array}$$

- f_i are *sigmoidal* functions.
- $\mathcal{C}$ is a *convex* set of constraints.

Sigmoidal programming is NP hard to approximate within a factor better than 16/17 (reduction from maxcut).

Algorithm

Branch and bound algorithm used to solve sigmoidal programming problem.

- Partition space into smaller regions $Q \in \mathcal{Q}$
- Compute upper and lower bounds

  $$L(Q) \le f^\star(Q) \le U(Q)$$

  on optimal function value

  $$f^\star(Q) = \max_{x \in Q \cap \mathcal{C}} \sum_i f_i(x_i)$$

  in region Q using convex optimization.
- Repeat until we zoom in on global max:

  $$\max_{Q \in \mathcal{Q}} L(Q) \le f^\star \le \max_{Q \in \mathcal{Q}} U(Q).$$

Example: auction

To maximize profit in auction bidding (in expectation), choose b by solving

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n (v_i - b_i) p_i(b_i) \\ \text{subject to} & b \in \mathcal{C}. \end{array}$$

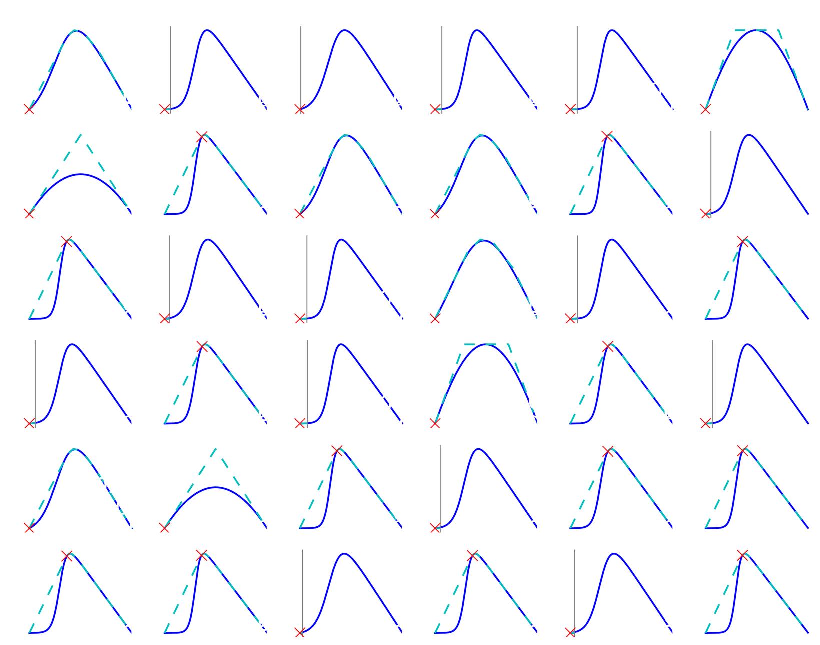
- Each i corresponds to an auction.
- v_i is the value of auction i.
- $p_i(b_i)$ is probability of winning auction i with bid b_i.
- $\mathcal{C}$ represents constraints on our bidding portfolio.

Convex constraints for auctions

- Minimum and maximum bids: $l \le x \le u$
- Budget constraint: $\sum_i x_i \le B$
- Sector constraint: $\sum_{i \in S} x_i \le B_S$
- Diversification constraint: $\forall |S| > k$,

  $$\sum_{i \in S} x_i \ge \epsilon \sum_i x_i$$
- And more (intersections are ok!)

e.g., solution with budget constraint



Example: election

To win the most votes (in expectation), choose x by solving

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n v_i f_i(x_i) \\ \text{subject to} & x \in \mathcal{C}. \end{array}$$

- Each i corresponds to a *constituency* (*e.g.*state, demographic, ideological group).
- v_i is # votes in constituency i (*e.g.*electoral, popular, etc).
- Politician panders an amount x_i to constituency i.
- $f_i(x_i)$ is expected vote share in constituency i.
- $\mathcal{C}$ represents constraints on what actions we are willing or able to take.

Convex constraints for politicians

- min, max position: $l \le y \le u$.
- max hrs in day: $\sum_i y_i \le B$.
- don't annoy any constituency too much: $w_i^T y \ge -\gamma$.

Solving the politician's problem

- Data from 2008 American National Election Survey (ANES)
- Respondents r rate candidates c as having positions $y^{rc} \in [1, 7]^m$ on m issues.
- For each candidate c and state i we construct a model to predict the likelihood of a respondent $r \in S_i$ in state i voting for candidate c as a function of the candidate's perceived positions y^{rc}.
- Suppose each state i has v_i votes, which they allocate entirely to the winner of the popular vote.
- y denotes the positions the politician takes on the issues.

Using our model,the politician's *pandering* to state i is given by $x_i = w_i^T y$, and the expected number of votes from state i is

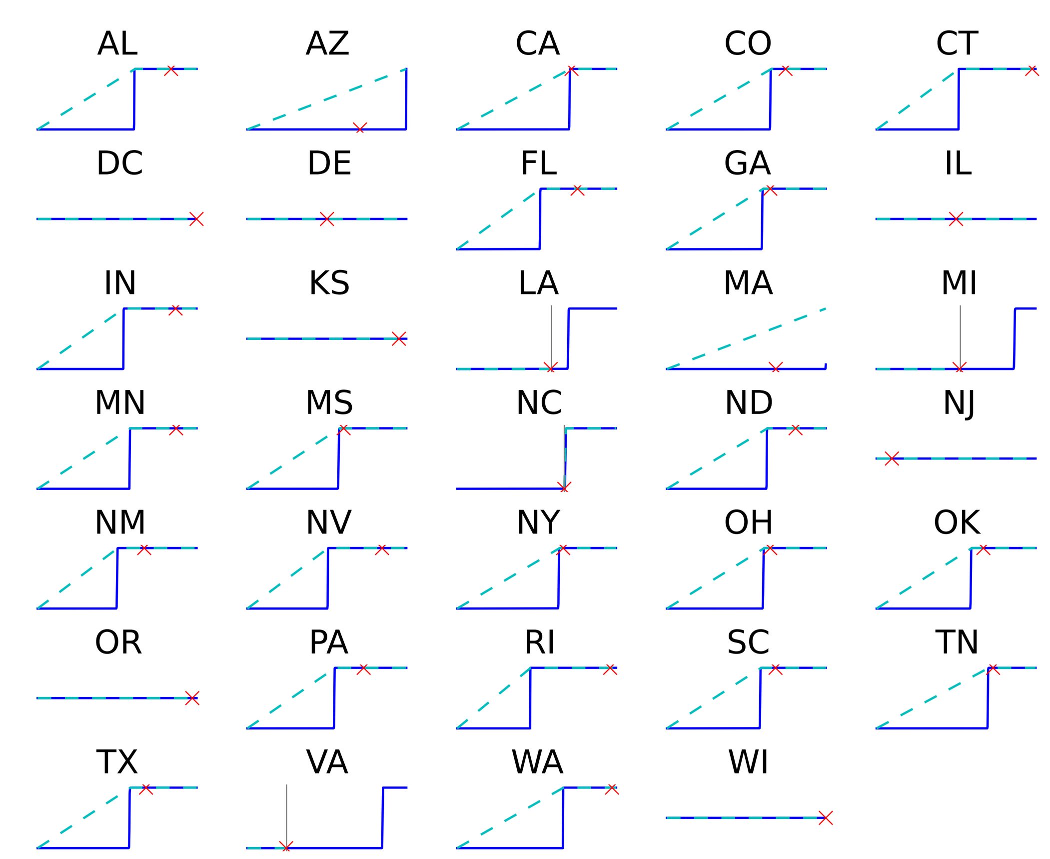
$$v_i \mathbf{1}(\mathrm{logistic}(x_i) > .5),$$

where $\mathbf{1}(x)$ is 1 if x is true and 0 otherwise. Hence the politician will win the most votes if y is chosen by solving

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n v_i \mathbf{1}(\mathrm{logistic}(x_i) > .5) \\ \text{subject to} & x_i = w_i^T y \quad \forall i \\ & 1 \le y \le 7. \end{array}$$

Results

Obama's optimal pandering:



Optimal positions for Obama

Issue	Optimal position	Previous position
Spending and Services	1.26	5.30
Defense spending	1.27	3.69
Liberal conservative	1.00	3.29
Govt assistance to blacks	1.00	3.12

Acknowledgement

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